On the spatial coherence of laser beams

1975 J. Phys. A: Math. Gen. 8 120

(http://iopscience.iop.org/0305-4470/8/1/018)

J. Phys. A: Math. Gen., Vol. 8, No. 1, 1975. Printed in Great Britain. © 1975

On the spatial coherence of laser beams

B E A Saleh and J M Minkowski

Electrical Engineering Department, The Johns Hopkins University, Baltimore, Maryland, USA

Received 5 December 1973, in final form 1 August 1974

Abstract. We examine the dependence of the spatial coherence across the beam of a multimode laser on the type of modes, the distribution of energy among them, and the degree of their statistical correlation.

1. Introduction

Laser beams are often assumed to be plane waves, lacking full temporal coherence, but having complete spatial coherence across their wavefronts. Since the spatial and temporal coherence properties of electromagnetic fields are coupled through the wave equations (Mandel and Wolf 1965), the above assumption is obviously an approximation. In this paper we investigate the validity of this approximation in practical lasers and analyse the situations under which the transverse coherence is impaired. We show that for practically occurring laser dimensions and bandwidths, a beam having a set of longitudinal modes with the same transverse indices (whether they are locked, unlocked or imperfectly locked, see, eg, Smith 1970), is approximately completely coherent. On the other hand, when a set of longitudinal modes having the same transverse indices interferes with another longitudinal set with different transverse indices, coherence can be incomplete. Morley *et al* (1967) have investigated this problem and obtained an expression for the spatial coherence distribution. In this expression, the contribution of the effect of interaction between modes having the same frequency has been overlooked. In this paper, we obtain a more general expression, based on a statistical treatment of the problem and in which we take into consideration the degree of statistical correlation of the modes. In particular, we study the case of two sets of TEM_{00} and TEM_{11} modes and show that the spatial coherence function depends not only on the distribution of energy among the modes but also on the degree of their statistical correlation (coupling).

The results of this work are valid regardless of the nature of physical processes taking place inside the laser (with which we are not concerned here).

2. Mutual coherence function

The second-order coherence of an optical field is described by the complex degree of coherence

$$\gamma(\boldsymbol{r}_1, \boldsymbol{r}_2; \tau) = \frac{\Gamma(\boldsymbol{r}_1, \boldsymbol{r}_2; \tau)}{\Gamma^{1/2}(\boldsymbol{r}_1, \boldsymbol{r}_1; 0)\Gamma^{1/2}(\boldsymbol{r}_2, \boldsymbol{r}_2; 0)} \tag{1}$$

where

$$\Gamma(\boldsymbol{r}_1, \boldsymbol{r}_2, \tau) = \langle V^*(\boldsymbol{r}_1, t) V(\boldsymbol{r}_2, t+\tau) \rangle \tag{2}$$

is the mutual coherence function (MCF) and $V(\boldsymbol{r}, t)$ is the electromagnetic field at the space–time point $(\boldsymbol{r}, t) = (x, y, z, t)$. The MCF can be determined either by solving the two wave equations which it has to satisfy under appropriate boundary and symmetry conditions, or by solving the wave equation for the field and then using equation (2). Since the wave equation has already been solved for different geometrical configurations of optical cavity resonators (Boyd and Gordon 1961, Kogelnik and Li 1966), it is simpler to follow the latter approach.

The solution is in general a superposition of many modes, each with deterministic spatial distribution but with an amplitude which is a complex random variable. Thus, the coherence function is completely describable by the covariance matrix of the amplitudes. Accounting for a finite bandwidth of each mode, the temporal dependence of each mode becomes a stochastic process and in general,

$$V(\boldsymbol{r}, t) = \sum_{nml} A_{nml} \int f_{nm}(\boldsymbol{r}, \omega) g_{nml}(\omega) \exp(-\mathrm{i}\omega t)\, \mathrm{d}\omega \tag{3}$$

where (n, m) and (l) are subscripts for transversal and longitudinal behaviour of a mode TEM_{nml}, $f_{nm}(\boldsymbol{r}, \omega)$ is a deterministic function describing the spatial behaviour of each mode at a specific frequency ω, $g_{nml}(\omega)$ is a complex random function describing the temporal statistics of the mode and A_{nml} is a complex random variable describing the amplitude of each mode.

In the case of confocal resonators (Boyd and Gordon 1961),

$$f_{nm}(x, y, z, \omega) = \frac{1}{(1+\xi^2)^{1/2}} H_m\left(X\left(\frac{2}{1+\xi^2}\right)^{1/2}\right) H_n\left(Y\left(\frac{2}{1+\xi^2}\right)^{1/2}\right) \exp\left(-\frac{\omega}{c}\frac{1}{b}\frac{\rho^2}{1+\xi^2}\right)$$
$$\times \exp -\mathrm{i}\left\{\frac{\omega}{c}\left(\frac{b}{2}(1+\xi) + \frac{\xi}{1+\xi^2}\frac{\rho^2}{b}\right) - (1+m+n)\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{1-\xi}{1+\xi}\right)\right]\right\} \tag{4}$$

where x and y are dimensions along the cross section, $\rho^2 = x^2+y^2$, $X = x(2\pi/a\lambda)$, $Y = y(2\pi/a\lambda)$, z is the longitudinal dimension measured from the centre of the cavity, b is the spacing between the mirrors, $\xi = 2z/b$, $2a$ is the diameter of the mirror, and $H_n(\cdot)$ is the nth Hermite polynomial. Also, $g_{nml}(\omega)$ is centred around the frequency

$$\omega_{nml} = \frac{\pi}{2}\frac{c}{b}(1+m+n) + \pi\frac{c}{b}l \tag{5}$$

where c is the velocity of light.

By substituting equations (3) and (4) into equations (2) and (1), we obtain the MCF in terms of the matrices $\langle A^*_{nml} A_{n'm'l'} \rangle$ and $\langle g^*_{nml}(\omega) g_{n'm'l'}(\omega) \rangle$. In the following we discuss special cases in which the results simplify.

3. Coherence of a beam in a single transverse mode

If the field is in a single mode and is monochromatic, ie, if $g_{nml}(\omega) = \delta(\omega - \omega_{nml})$, then $|\gamma| = 1$ and the field is completely coherent, as expected.

If the field is in a single mode, but is quasi-monochromatic with a bandwidth $\Delta\omega$ and central frequency ω_0, then it can be easily shown that under the conditions:

$$\frac{\Delta\omega}{\omega_0}\frac{\rho^2}{by}\frac{1}{1+\xi^2} \ll 1 \tag{6a}$$

and

$$\frac{\Delta\omega}{\omega_0}\eta\left|1-\frac{1}{2\sqrt{\eta}}\frac{H_{\nu+1}(\sqrt{\eta})}{H_\nu(\sqrt{\eta})}\right| \ll 1 \qquad \nu = n, m \tag{6b}$$

where

$$\eta = \frac{2\pi}{a\lambda}\frac{2}{1+\xi^2}\rho^2, \tag{7}$$

the spatial distribution function $f^*_{nm}(x_1, y_1, z, \omega)$. $f_{nm}(x_2, y_2, z, \omega')$ is approximately insensitive to the variations of ω and ω' within the bandwidth $\Delta\omega$ and the field is almost completely coherent.

To show this more clearly we calculated the transversal coherence function for the fundamental mode $n = m = 0$ when the field is stationary and Lorentzian (ie, $\langle g^*_{0,0}(\omega)g_{0,0}(\omega)\rangle$ is Lorentzian) with a bandwidth $\Delta\omega$ and centre frequency ω_0. The degree of coherence between a point on the beam axis and an off axis point is

$$|\gamma(\rho, 0; \tau)| = \exp\left[-\Delta\omega\left|\tau-\frac{\xi}{\omega_0}\left(\frac{\rho}{\rho_0}\right)^2\right|\right] \tag{8}$$

where $\rho_0 = [(\lambda b/2\pi)(1+\xi^2)]^{1/2}$ is the radius of the beam ($1/e^2$ point of the Gaussian intensity distribution).

Thus, the degree of coherence drops to $1/e$ of its value at a radius of coherence

$$\rho_c = \rho_0\left(\frac{\omega_0}{\Delta\omega}\frac{1}{\xi}\right)^{1/2} = \left(\frac{\omega_0}{\Delta\omega}\right)^{1/2}\left(\frac{\lambda b}{2\pi}\right)^{1/2}\left(\frac{b}{2z}+\frac{2z}{b}\right)^{1/2}$$

which has a minimum when $z = \frac{1}{2}b$, ie, at the reflector surface and is equal to $[(\lambda b/\pi)(\omega_0/\Delta\omega)]^{1/2}$. For all practical cases this radius is very large and hence the beam is almost completely coherent.

In the case when many longitudinal modes coexist, all with the same transverse indices (m and n) and under the same conditions as in equation (6) (with $\Delta\omega$ the total bandwidth of all modes and ω_0 their central frequency), the resulting field is transversely approximately completely coherent, irrespective of the relationship between the longitudinal modes, ie, whether they are free, mode locked or partially mode locked. This conclusion holds for both stationary and non-stationary (pulsed) fields. Note that the above analysis is independent of the physics of the lasing phenomenon and holds even when the laser is operated far below the threshold and radiates as a thermal source. Kimble and Mandel (1973) have recently confirmed experimentally that in this case the field of the beam is indeed nearly coherent.

4. Coherence of multi-mode beams

In this section we consider the only case in which the transverse coherence is considerably reduced, namely when more than one transverse mode coexist. For simplicity, we assume that each of the modes is monochromatic. Using equation (4), we get

$$\Gamma(\boldsymbol{r}_1, \boldsymbol{r}_2; 0) = \sum_{nml} \sum_{n'm'l'} \langle A^*_{nml} A_{n'm'l'} \rangle f^*_{nm}(\boldsymbol{r}_1, \omega_{nml}) \times f_{n'm'}(\boldsymbol{r}_2, \omega_{n'm'l'}) \langle \exp[\mathrm{i}(\omega_{nml} - \omega_{n'm'l'})t] \rangle .$$

Realizing that

$$\langle \exp[\mathrm{i}(\omega_{nml} - \omega_{n'm'l'})t] \rangle = 0,$$

for $\omega_{nml} \neq \omega_{n'm'l'}$, we obtain

$$\Gamma(\boldsymbol{r}_1, \boldsymbol{r}_2; 0) = \sum_{nml} \langle |A_{nml}|^2 \rangle f^*_{nm}(\boldsymbol{r}_1, \omega_{nml}) f_{nm}(\boldsymbol{r}_2, \omega_{nml}) + \sum_{nml} \sum_{n'm'l'} \langle A^*_{nml} A_{n'm'l'} \rangle f^*_{nm}(\boldsymbol{r}_1, \omega_{nml}) f_{n'm'}(\boldsymbol{r}_2, \omega_{nml}). \tag{9}$$

$$\omega_{nml} = \omega_{n'm'l'} .$$

The expression of the MCF obtained by Morley *et al* (1967) is similar to equation (9) but without the second term. In their special example, only two transverse modes exist, TEM_{00} and TEM_{10}, and the second term of equation (9) vanishes since it is impossible to have $\omega_{00l} = \omega_{10l'}$, $l \neq l'$. This is not always valid. For example, the two modes TEM_{00} and TEM_{11} can have $\omega_{00l} = \omega_{11l'}$ when $l = l' + 1$. In this case, equation (9) becomes

$$\begin{aligned}\Gamma(\boldsymbol{r}_1, \boldsymbol{r}_2, 0) = {} & \sum_l \langle |A_{00l}|^2 \rangle f^*_{00}(\boldsymbol{r}_1, \omega_{00l}) f_{00}(\boldsymbol{r}_2, \omega_{00l}) \\ & + \langle |A_{11l}|^2 \rangle f^*_{11}(\boldsymbol{r}_1, \omega_{11l}) f_{11}(\boldsymbol{r}_2, \omega_{11l}) \\ & + \langle A^*_{00l} A_{11,l-1} \rangle f^*_{00}(\boldsymbol{r}_1, \omega_{00l}) f_{11}(\boldsymbol{r}_2, \omega_{00,l}) \\ & + \langle A_{00l} A^*_{11,l-1} \rangle f_{00}(\boldsymbol{r}_2, \omega_{00l}) f^*_{11}(\boldsymbol{r}_1, \omega_{00l}).\end{aligned}$$

By neglecting the dependence of the functions f on the frequency, and using the normalization $\Sigma_l \langle |A_{00l}|^2 \rangle = 1$, we can write

$$\begin{aligned}\Gamma(\boldsymbol{r}_1, \boldsymbol{r}_2, 0) = {} & f^*(\boldsymbol{r}_1, \omega_0) f_0(\boldsymbol{r}_2, \omega_0) + \delta f^*_{11}(\boldsymbol{r}_1, \omega_0) f_{11}(\boldsymbol{r}_2, \omega_0) \\ & + \epsilon f^*_{00}(\boldsymbol{r}_1, \omega_0) f_{11}(\boldsymbol{r}_2, \omega_0) + \epsilon^* f_{00}(\boldsymbol{r}_2, \omega_0) f^*_{11}(\boldsymbol{r}_1, \omega_0)\end{aligned}$$

where $\delta = \Sigma_l \langle |A_{11l}|^2 \rangle$ is a real number representing the ratio of the average intensities of the two sets of modes, $\epsilon = \Sigma_l \langle A^*_{00l} A_{11,l-1} \rangle$ is a complex number representing their average correlation, and ω_0 is the central frequency. The two parameters δ and ϵ describe completely the intensity distribution and the MCF of the laser beam. They satisfy the inequality $|\epsilon|^2 \leqslant |\delta|$. When $\epsilon = 0$, the two modes are statistically uncorrelated, and when $|\epsilon|^2 = |\delta|$ the two modes are completely correlated (mode locked).

By simple substitution, the degree of transverse coherence between a point at the beam centre and an off-axis point, and the normalized intensity distribution are given in equations (10) and (11) respectively

$$|\gamma(0, 0, z; x, y, z; 0)| = \left(\frac{1 - 2|\epsilon| U \sin \Delta + |\epsilon|^2 U^2}{1 - 2|\epsilon| U \sin \Delta + \delta U^2} \right)^{1/2} \tag{10}$$

$$J(x, y, z) = \frac{\Gamma(x, y, z; x, y, z; 0)}{\Gamma(0, 0, z; 0, 0, z; 0)} = (1 - 2U|\epsilon| \sin \Delta + \delta U^2) \exp -2 \left(\frac{X^2 + Y^2}{1 + \xi^2} \right) \tag{11}$$

where

$$U = \frac{8XY}{1+\xi^2}$$

and

$$\Delta = \angle\epsilon - 2\tan^{-1}\frac{1-\xi}{1+\xi}$$

in which $\angle\epsilon$ is the phase of the parameter ϵ.

We plot these equations in figures 1 and 2. Figure 1 shows the intensity distribution for different values of δ when the two modes are uncorrelated ($\epsilon = 0$, full curves). It also shows how the intensity distribution changes drastically if the modes are correlated (broken curves). The result is very sensitive to the phase of the correlation parameter ϵ. Figure 2 shows the drop in the degree of transverse coherence for different ratios of intensities of the two uncorrelated modes (full curves). The degree of coherence is also quite sensitive to the correlation phase between the two modes (broken curves). When the two modes are completely correlated ($|\epsilon|^2 = \delta$) the degree of coherence is 1, irrespective of the correlation phase.

5. Conclusions

The above examples demonstrate that the degree of spatial coherence depends on the types of modes, the distribution of energy among them, and the degree of their statistical

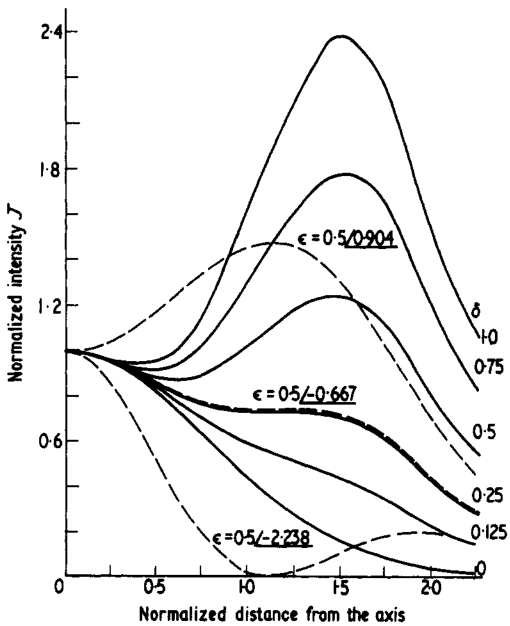

Figure 1. Variation of normalized intensity with distance from the beam axis for different values δ and ϵ. (Distance from the median of the cavity $\xi = 2$.) Full curves represent the uncorrelated case ($\epsilon = 0$). Broken curves represent the case of a set of partially correlated modes with the indicated ϵ and $\delta = 0{\cdot}25$.

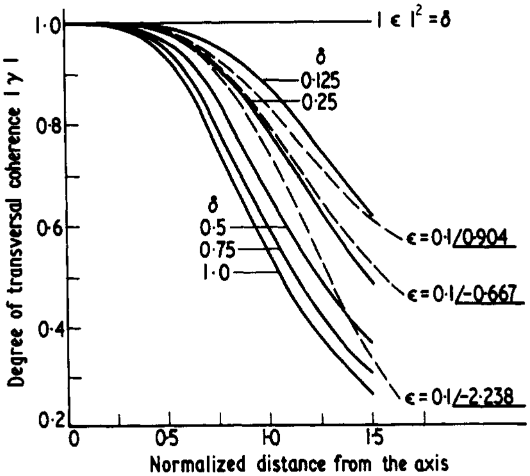

Figure 2. Variation of the transverse degree of coherence with the distance of its two points (one on the axis), for different values δ and ϵ ($\xi = 2$). Full curves represent the uncorrelated case ($\epsilon = 0$). Broken curves represent a set of partially correlated modes with indicated values of ϵ and $\delta = 0{\cdot}25$.

coupling. Moreover, the theoretical graphs of figures 1 and 2 show that it is, in principle, possible to determine the parameters δ and ϵ from measurements of the intensity and the MC distributions†. The determination of these parameters is of interest since this sheds light on the physical processes taking place inside the laser. The feasibility of such measurements has been demonstrated by several workers (Bertolotti *et al* 1964, Morley *et al* 1967, Kimble and Mandel 1973).

References

Bertolotti M, Diano B and Sette D 1964 *Nuovo Cim.* **33** 1705–9
Boyd G D and Gordon J P 1961 *Bell. Syst. Tech. J.* **40** 489–508
Kimble H J and Mandel L 1973 *J. Opt. Soc. Am.* **63** 1550–1
Kogelnik H and Li T 1966 *Appl. Opt.* **5** 1550–67
Mandel L and Wolf E 1965 *Rev. Mod. Phys.* **37** 231–87
Morley D C, Schofield D G, Allen L and Jones D G 1967 *Brit. J. Appl. Phys.* **18** 1419–22
Smith P 1970 *Proc. IEEE* **58** 1342–57

† Such determination is unique provided the measurements extend over a sufficient range of distances of the two space points (ie over a range of values of the 'normalized distance from the axis' in figures 1 and 2).